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# Uni-dimensional models of coalition formation: non-existence of stable partitions

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**Abstract:** Consider a finite population located along the real line, which generates a demand for public facilities. The set-up cost of each facility is the same and is given by a positive constant. In addition to contributing towards the facility cost, every user bears transportation cost to the facility location. These are classical prerequisites for the Uncapacitated Facility Location Problem; however our focus is on game-theoretic aspects of the problem. Assuming for simplicity that the set-up cost of facilities are equally shared among its members, we examine the existence of a ''stable'' set of facilities (or, equivalently, a partition of the set of players) such that no coalition (i. e. a nonempty subset of players) can set up a new facility that would reduce the total cost incurred by each member of the coalition. The simple majority condition requires that every group places the facility at the location of its *median* resident. In general, however, a median voter is not uniquely defined. This paper offers a universal counterexample that regardless of the selection of a median resident, a stable partition of individuals into users of various facilities fails to exist.

**Keywords:** Set of partitions, median location, coalitional stability, equal-share rule, median choice selection

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## 1. Introduction

The paper examines an application of the well-known Uncapacitated Facility Location Problem, abbreviated UFLP [5], to the game-theoretical investigation of stability and efficiency of societal organization. To briefly describe a UFLP, consider a set of players $N = \{1, \ldots, n\}$ and a set of possible facility locations $X$, that, for simplicity, is assumed to coincide with the real line. The location of player $i \in N$ is denoted by $x_i \in X$. Every player should belong to one and only one facility, the cost of which is given by a positive constant $g$. There are also individual assignment (transportation) costs so that the player $i \in N$ assigned to facility $k \in X$ bears the cost $c_{i,k}$. A variant of UFLP in this setting is reduced to identifying a nonempty subset $K \subset X$ and assignment $f : N \to K$ in order to minimize the total cost

$$g \cdot |K| + \sum_{i \in N} c_{i,f(i)}, \tag{1}$$

where $|K|$ stands for the cardinality of the set $K$.

The assignment part of the problem is straightforward, as every player $i$ should be assigned to the facility $k$ that minimizes his individual cost $c_{i,k}$. Finding an optimal subset of facilities is, however, a complicated problem that has been under investigation in the literature for a long period of time.

The game-theoretical angle here is to examine whether the choice of facilities is stable in the sense that no group of players $S \subset N$ can be offered a different facility that would benefit every member of $S$. However, in order to provide a meaningful answer to the stability question, the UFLP does not go far enough. In order to evaluate their potential benefits of switching to another location, the players need to know the allocation of the location cost $g$ among the users of the facility. Namely, there is a contribution scheme that assigns a cost share to every user of the facility. Let $S_k$ be the set of players assigned to the facility $k$, that is $S_k = \{i \in N | f(i) = k\}$. Then a *balanced* contribution scheme $t : S_k \to \mathbb{R}_+$ should satisfy

$$\sum_{i: i \in S_k} t^i_{S_k} = g. \tag{2}$$

The stability in the case with no further restrictions on balanced contribution schemes has been examined in [10]. The literature ([3]) has also extensively examined the *Rawlsian* (or full-equalization) contribution scheme $r$, where the sum of the facility cost and the individual costs is shared equally among all users of the facility. That is,

$$r^i_{S_k} = \frac{g + \sum_{i: i \in S_k} c_{i,k}}{|S_k|} - c_{i,k}. \tag{3}$$

The most important class of contribution schemes that has been studied in the literature and is the subject of this paper, is *equal share* (no equalization) allocations:

$$e^i_{S_k} = \frac{g}{|S_k|}. \tag{4}$$

More specifically, in the framework considered in this paper, a player $i$ who is assigned to the facility $k$ incurs the total cost of $e^i_{S_k} + c_{i,k}$, where is $c_{i,k}$ is the transportation cost of $i$ to the location of facility $k$. The location of facility is determined on the basis of the MAT (Minimal Aggregate Transportation) principle, where the group $S_k$ determines the location by minimizing the total sum

$$\sum_{i \in S_k} c_{i,k}.$$

And here comes the problem that is the main topic of the paper. Even under the uni-dimensional case when all the facilities are located on the straight line, the distance-minimizing location for linear transportation costs may not be uniquely determined in the case of even-sized groups. Indeed, taken two players located in different points, *any* location on the interval connecting these two points would minimize the total linear cost. Thus, in order to predict players' behavior, one has to assume an arbitrary choice of cost-minimizing locations as it is done in [4]. For example, it could be a middle point, left-most, or rightmost point within the interval of cost-minimizing choices.

This paper resolves the arbitrariness issue. Namely, it presents an example that shows that even in the case where threats to stability are restricted to groups for which a cost-minimizing location is unique, a stable configuration structure of facilities and assignments fails to exist.

## 2. A universal counterexample: motivation and description

We now offer the definition that formalizes the UFLP game-theoretical aspect relevant to this paper.

Definition 1. *A partition* $\pi = \{S_1, \ldots, S_d\}$, *where* $N = S_1 \sqcup \ldots \sqcup S_d$, *is called* stable (in the coalition sense) *under the equal-share rule*, *if for every coalition* $S$ *there* *exists a player $i \in S$ for which the following inequality holds*:

$$\frac{g}{|S|} + c_{i,k(S)} > \frac{g}{|S^i|} + c_{i,k^i}, \qquad (5)$$

*where $S^i \in \pi$ is the coalition in $\pi$ that contains $i$. Here, $k(S)$, location of the facility in $S$, and $k^i$, location of the facility in $S^i$ are chosen on the basis of the MAT principle. In what follows, we call any such choice a* median location. *In our uni-dimensional setting, we use the Euclidean distance, i. e., $c_{i,k} = |i-k|$ for individual $i$ and location $k$.*

In [4], a counter-example is given for which no stable partitions exist: $x_1 = x_2 = x_3 = 0$; $x_4 = x_5 = 19/60$, where $g = 1$. Graphically,

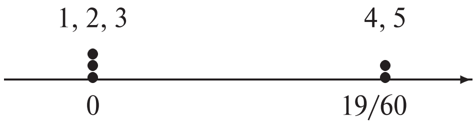


**Figure 1.** A counter-example to coalitional stability on the real line

Given this counter-example, a general failure of stability is obviously reinforced for more general classes of the UFLP environments. This is in contrast to the *TU* cost sharing rule, for which a stable partition always exists in the uni-dimensional UFLP settings, [8].

This above counter-example is built ''on a margin'', for very special (though of a positive measure) combinations of parameters. Also, it allows a coalition to choose any of its median locations for its facility. But what if we impose a modified equal-share cost sharing rule, in which, in case when there are multiple medians, the facility location is chosen at the central point of the segment of medians, a ''focal point'' for that matter? Can this hedonic modification of the equal-share rule preserve stability?

In [4], it is shown that even in this case the answer is "No". There still exists a counter-example, this time containing 8 players, and more complex in its structure. Here it is (in the figure, $\delta$ is chosen to be a sufficiently small positive number):


**Figure 2.** A counter-example to coalitional stability for the central median equal-share cost sharing rule

However, the latter example is still very special, even more than the one before that. The question then naturally arises: Is there still a possibility to identify a median specification under the unrestricted equal-share cost sharing rule, such that coalitional stability will be guaranteed for the entire class of uni-dimentional UFLP settings? This would be a very desirable choice mechanism in our context.

The main result of this paper rules out such a possibility. Moreover, it says that even *preferential* treatment of groups in a partition, versus coalitions posing secession threats, cannot help in providing a general stability result. In fact, there exists an environment of uni-dimentional UFLP such that, for *any* partition of individuals into facilities users and an *arbitrary* selection of median facilities' locations, one can find a coalition with the unique median location that reduces the total cost of each of its members. (Note that the requirement of a unqe median location restricts the range of feasible coalitions to those with an odd number of players and those for which two median players have an identical location). In other words, for any partition and any assignment of median locations there is always a coalition with a unique median that is beneficial for all its members in terms of incurred total cost.

Specifically, consider the set of players, $N = \{1,2,3,4,5,6,7,8,9,10,11,12,13\}$, with $g = 120$ and

$$x_1 = x_2 = x_3 = x_4 = x_5 = x_6 = x_7 = -4; \quad x_8 = 0;$$

$$x_9 = x_{10} = x_{11} = x_{12} = x_{13} = 12.$$

The corresponding graphical representation looks as follows:


**Figure 3.** A universal uni-dimentional counter-example to coalitional stability

**The "give up your hope" theorem.** Given the data presented above, for *any* partition $\pi = \{S_1, \ldots, S_L\}$ of the set of players

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\},$$

and *any* selection of median locations $\{k_l\}_{l=1,\dots,L}$ for corresponding groups, there exists a *group $T$ with the unique median location $k$* such that the inequality

$$c(i,T) = \frac{1}{|T|} + |x_i - k| < c(i,S^i) = \frac{1}{|S^i|} + |x_i - k^i| \tag{6}$$

holds for each member $i \in T$. To recall, $S^i \in \pi$ is the group of users that contains $i$ and $k^i$ is its facility location.

## 3. Proof of the main result

The **proof** is both tedious and technical, but I present it here, in order to demonstrate some useful methods that could be applicable for a further work in this field.

Suppose, on the contrary, that there exists a stable partition $\pi = \{S_1, \dots, S_Q\}$ of the set of players

$$N = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\},$$

taken together with a set of medians $\{k_q\}_{q=1,\dots,Q}$ within the corresponding groups. Let us prove some preliminary results first.

Denote by $\mathcal{L} = \{1, \dots, 7\}$ the set of individuals whose locations are on left at $l = -4$, and by $\mathcal{R} = \{9, \dots, 13\}$ the set of individuals whose locations are on right at $r = 12$.

Proposition $A$. *There is only one group in $\pi$ whose median location is at $l$. Moreover, members of $\mathcal{L}$ constitute a majority in this group.*

Proof. Indeed, let us split each group in $\pi$ into two parts: those from $\mathcal{L}$, and all others. Since 7 players are from $\mathcal{L}$ and 6 are not, there exists at least one group where $\mathcal{L}$ citizens are in majority.

Let us denote this group, whose median location is at $l$, by $S_l$. Let us prove that such a group is unique. If there were another group with a median location at $l$, merging it with group $S_l$ would preserve the majority of $\mathcal{L}$ members, the transportation costs would also remain the same but all the monetary costs would decrease. It is a contradiction to the stability of partition $\pi$ as the players in the newly formed group with a unique median location would incur lower total costs.

That is why there is only one group in $\pi$ with a median location in $l$ and the members of $\mathcal{L}$ are in majority there. $\square$

Proposition $B$. *There is a group in $\pi$ that contains at least 7 players.*

Proof. Indeed, if all groups are small (no more than 6 players) then every player's costs are at least $120/6 = 20$. In particular, this is true for members of $\mathcal{L}$ who can constitute their own seven-member group and reduce their costs to $120/7 < 120/6$, again contradicting the stability of partition $\pi$. □

We can strengthen the assertion of Proposition B:

Proposition $B'$. *There is a group in $\pi$ that contains at least 8 players.*

Proof. Suppose, on the contrary, that there is no 8-players group in $\pi$. Due to Proposition B, the largest group in $\pi$ contains 7 players. Since $\pi$ is a stable partition, the group $S = \{x_1, \dots, x_8\}$ poses no threat for $\pi$ (note that $S$ has a unique median).

But if this group forms, its first seven players would have costs equal to $120/8 = 15$ (which is less than $120/7$ — the minimal cost for any player in $\pi$).

Thus, the eighth player in $S$ must incur higher costs than he has in $\pi$. His costs in $S$ are $120/8 + 4 = 19$, lower than in any coalition of no more than six players: $19 < 120/6 = 20$. That is why individual 8 is a member of the largest coalition. But, in addition, he must be the median player in the largest group! Indeed, a group of 7 players has a unique median and it must coincide with the location of its median player. If player 8 is not median, he has to bear transportation costs of at least 4 (the distance to the closest point where the median may be). But $120/7 + 4 > 19$ — and he surely would join $S$ unless he is a median in his group in $\pi$.

Denote the coalition that contains 8 by $W$. Since player 8 is the median of $W$, it must contain 3 players from $\mathcal{L}$ and 3 players from $\mathcal{R}$. Consider then $\mathcal{R}$ that contains three members of $W$. The costs of 3 citizens of $W$ are $120/7 + 12 > 120/5 = 24$. The other two members of $\mathcal{R}$ either participate in a 4-member group, or in a group with 5 or 6 players with the facility located at $l$. In the former case their costs are at least $120/4 = 30 > 24$, in the latter — not less than $120/6 + 16 = 36 > 24$. In any case, their costs are higher than $120/5 = 24$. Thus, the emergence of the coalition $\mathcal{R}$ rules out the stability of partition $\pi$. □

Let us denote the largest group in $\pi$ by $V$.

Lemma 1. *If $V = S_l$ (i.e. the largest group has the facility at $l$), then*

$$1, 2, \dots, 7, 8 \in V.$$

PROOF. Since $|V| \geqslant 8$, if there are $\mathcal{L}$-residents who are not members of $V$, their costs are at least $120/5 = 24$ (there are no more than 5 players outside of $V$). But if they join $V$, the median of the new coalition would remain at $l$ and all members of $V$ and the newcomers would be better off (the new members would bear costs no more than $120/9 < 24$). That is, the new extended coalition would contradict the assumed stability of $\pi$.

So, $1, 2, \dots, 7 \in V$. In this case any potential extension of $V$ would not alter its median and is therefore admissible.

Let us turn to player 8. If he is not in $V$, he bears costs not lower than than $120/5 = 24$ (there are only 5 players left!), if he joins $V$ he would incur costs no more than $120/9 + 4 < 24$ and all other members of $V$ would pay less. Thus, the extension of $V$ violates the stability of partition $\pi$ and player 8 has to be in $V$. □

In the following three propositions the minimal size of the largest group is increased to 11.

PROPOSITION $C$. *The largest group cannot consist of* 8 *members.*

PROOF. Suppose the contrary. Let $|V| = 8$. Then there are 3 variants for locating the median $k$ in $V$:

$C8l$: $k = l$,

$C8r$: $k = r$,

$C8c$: the median $k$ is neither $l$ nor $r$.

$C8l$: In this case it immediately follows from Lemma 1 that $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Then it must be the case that $\pi = \{V, [9, 10, 11, 12, 13]\}$ (in any other partition 5 members of $\mathcal{R}$, having a unique median, can join together and reduce their costs).

This partition $\pi$ can be threatened by the following group:

$$D = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}.$$

Since the median of $D$ is at the player 8 location $x_8 = 0$, he is obviously better off.

The residents of $\mathcal{R}$ who are members of $D$ are better off:

$$\frac{120}{5} = 24 > \frac{120}{11} + 12.$$

The residents of $\mathcal{L}$ in $D$ who are members of $D$ are also better off:

$$\frac{120}{8} = 15 > \frac{120}{11} + 4.$$

Sad, but true: the two players left behind (1 and 2) have to pay 60 each — the laws of coalitional game theory are sometimes cruel!

$C8r$: In this case the costs of (at least) two $\mathcal{L}$-residents in $V$ are equal to $120/8 + 16 = 31 > 120/7$, while the remaining residents of $\mathcal{L}$ are in coalition of size no more than 5 and therefore pay at least $120/5 = 24$. It immidiately follows that group $\mathcal{L}$ destroys the stability of $\pi$.

$C8c$: The group $V$'s median (denote it by $k$) is located somewhere between $l$ and $r$. Denote by $\rho$ the distance from $l$ to $k$, so that $k = \rho - 4$. Let us note that there are no more than 4 citizens from $\mathcal{R}$ and from $\mathcal{L}$ in $V$: so, one of the following distributions of members of $V$ across three types $\mathcal{L}, 8, \mathcal{R}$ is possible: $\{4, 0, 4\}$, $\{4, 1, 3\}$, and $\{3, 1, 4\}$.

The residents of $\mathcal{L}$ who are not in $V$ bear costs no less than $120/5 = 24 > 120/7$, and the $\mathcal{R}$ residents who are not in $V$ bear costs greater than $120/5 = 24$ (either they are in small coalition or incur high transportion costs).

To guarantee the coalitional stability of $\pi$ we need to choose the median point in such a way that both $\mathcal{L}$ and $\mathcal{R}$ members of $V$ incur sufficiently low costs at $\pi$. More specifically, the total cost of $\mathcal{L}$ members of $V$ should not exceed 120/7 (otherwise, the group $\mathcal{L}$ would emerge) and the total cost of $\mathcal{R}$ members of $V$ should not exceed 120/5 (otherwise, the group $\mathcal{R}$ would emerge). That is, the following inequalities hold

$$15 + \rho \leqslant \frac{120}{7}, \quad 15 + 16 - \rho \leqslant 24 = \frac{120}{5}.$$

By summing them up, we obtain that $46 \leqslant 24 + 120/7 < 42$, a contradiction that shows that any stable partition has a coalition of at least 9 players. □

Let us prove an additional lemma:

LEMMA 2. *Coalition $V$ cannot coincide with $S_l$, i. e. the group with the facility at $l$.*

PROOF. Suppose the contrary, i. e., let $V = S_l$. Then we have $1, 2, 3, 4, 5, 6, 7, 8 \in V$. Without loss of generality we may assume that

$$V = \{1, 2, 3, 4, 5, 6, 7, 8, \ldots, m\},$$

where $m = |V|$, and as above it can be easily shown that

$$\pi = \{V; [m+1, \dots, 13]\}$$

if $m \leqslant 12$ and $\pi = \{V\}$ if $m = 13$. Consider these two possibilities.

Suppose that $9 \leqslant m \leqslant 12$. Then the costs of players outside of $V$ are at least $120/4 = 30$. All of these players are from $\mathcal{R}$ and let them join $V$ and unite everybody in the grand coalition $N$. Note that the center of $N$ remains at $l$, and all $V$ members are better of (since their monetary costs decrease and transport costs remain the same). The joining players also reduce their costs: $120/13 + 16 < 30$.

Thus, the only possibility left is $\pi = \{V\}$. But this is also impossible: the set $\mathcal{R}$ would yield the costs lower than in the grand coalition, which are

$$\frac{120}{13} + 16 > \frac{120}{5} = 24. \qquad \square$$

Proposition $C9$. *Coalition $V$ has at least* 10 *members.*

Proof. Once again, suppose the contrary. Let $|V| = 9$, and examine the following 3 options for the median location $k$ for $V$:

$$C9l: \quad k = l,$$
$$C9r: \quad k = r,$$
$$C9c: \quad k = 0.$$

$C9l$: Lemma 2 implies that it is impossible.

$C9r$: In this case recall that the set $\mathcal{L}$ yields the cost $120/7$ to its members. At the same time, the residents of $\mathcal{L}$ in $V$ bear costs equal to $120/9 + 16 > 120/7$, and those not in $V$ bear costs at least of $120/4 = 30 > 120/7$. So, this case is also impossible.

Only $C9c$ remains. The only possible player distribution here is $\{4, 1, 4\}$: 4 citizens from $\mathcal{L}$, player 8, and 4 citizens from $\mathcal{R}$.

Here a new trick is needed: The new coalition $Q$ is created by adding to $V$ one player from $\mathcal{R}$ and one from $\mathcal{L}$. The new coalition has a unique median and, therefore, is admissible. Moreover, its median remains in 0. Hence, all players from $V$ are better off and the joining players have to pay not more than

$$\frac{120}{11} + 12 < 30 = \frac{120}{4},$$

the minimum they had to pay before. The coalition $Q$ is an admissible threat to partition $\pi$, so it is impossible that $|V| = 9$. □

PROPOSITION $C10$–$11$. *Coalition $V$ contains at least 12 players.*

PROOF. As usual, suppose the contrary. Lemma 2 deals with the case where the group facility is located in $\mathcal{L}$. Here we turn to other possibilities that would imply that there are no more than 5 residents of $\mathcal{L}$ in $V$ (otherwise the facility would be at $l$!)

Hence, at least two residents of $\mathcal{L}$ exist who are not in $V$. Their costs are very high: no less than $120/3 = 40$ (since there are no more than 3 players outside $V$).

Now we are going to use a new, but fundamental trick, which is highly counter-intuitive. Take a nonconsecutive coalition as a threat for stability of $\pi$: of two $\mathcal{L}$ residents and all residents of $\mathcal{R}$ — without the player in the middle! The median of this coalition is $r$, and it is, therefore, admissible. The two "losers" from $\mathcal{L}$ face costs so high that they would agree to join almost any group, the following inequality shows that this is indeed the case: $120/7 + 16 < 40$.

The only remaining thing is to check whether the costs of $\mathcal{R}$ residents indeed decline. Moreover, we need to check only $\mathcal{R}$ residents in $V$ — those not in $V$ obviously would agree to join the new coalition.

The new costs of $\mathcal{R}$ members equal $120/7$ (they bear only monetary costs). The initial costs (in $\pi$) of $\mathcal{R}$ members equal $120/q + \rho$, where $q = |V| = 10$ or $11$, and $\rho$ is the distance from $r$ to the center of $V$ coalition.

The case of $q = 11$ is now rather obvious. Indeed, without loss of generality $V$ is the following group:

$$\{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$$

(only two members of $\mathcal{L}$ are left out). But it has median at point 0, i. e. $r = 12$. The $\mathcal{R}$ members costs equal to $120/11 + 12$, which is higher than $120/7$. Therefore partition $\pi$ is not stable.

Only the case of $q = 10$ remains. Here the situation is a bit tricky, but still tractable. We need to recall that the coalition $\mathcal{L}$ may also pose a threat to $\pi$. In case of secession its participants bear a cost of $120/7$ each. It is clear that $\mathcal{L}$ citizens not in $V$ would join this coalition (with their minimum costs of 40!), so we turn to $\mathcal{L}$ citizens in $V$.

The partition $\pi$ would be stable only if both threats above could not be carried out. Again, let $\rho$ to denote a distance from the facility in $V$ to $r$, so $16 - \rho$ is the distance from $l$ to the $V$ facility. Hence, the following two inequalities must hold (note that $q = 10$):

$$\frac{120}{10} + \rho \leqslant \frac{120}{7}, \quad \frac{120}{10} + 16 - \rho \leqslant \frac{120}{7}.$$

Summing them up, we get that the inequality

$$12 + 12 + 16 = 40 \leqslant \frac{240}{7} < 35$$

must hold, and obtain a contradiction! □

There is only one statement left to prove:

Proposition $C12$. *There is no stable partition such that* $|V| = 12$.

Proof. It immediately follows from Lemma 2 that the only "dropout" resides in $\mathcal{L}$. Note that the facility in $V$ cannot be located to the right of the middle player — otherwise the median condition is broken. That is why the distance from $r$ to the facility in $V$ center is at least 12.

Hence the members of $\mathcal{R}$ (all of them are in $V$) incur costs which are at least $120/12 + 12 = 22$.

Let us use the same trick as above: all $\mathcal{R}$ members would unite with the "dropout" from $\mathcal{L}$. The median of this new coalition is at $r$, hence, this coalition is admissible. The "dropout" would gladly agree to join any coalition (with initial costs of 120!), his costs equal $120/6 + 16 = 36$. And the members of $\mathcal{R}$ would incur the costs of $120/6 = 20$ each, which is lower than 22 — their costs in $V$. Hence the partition $\pi$ cannot be stable! □

The universal counterexample is constructed and the proof of the theorem is complete!

## 4. Notes on relevant research and bibliography

A good survey of the Uncapacitated Facility Location Problem is [5]. The idea to raise game-theoretic questions within the UFLP environment comes back to [1], on the one hand, and to the branch of the operations research literature, on the other. The latter deals with the Transferable Utility cooperative game where the cost of a given coalition coincides with its total cost when running the facility for its members in an optimal location, the solution concept being that of a *core of a coalition partition form* by [2].

Relevant references of this approach are [11, 13], and [8]. In the latter paper, it is proved that the UFLP game constructed above, being considered on the real line, always has a nonempty core. A nice characterization of the core in the uni-dimensional uniform distribution case is due to [6]; as for the two-dimentional case, the core is empty [7].

However these all are considerations within the TU game, when arbitrary cost redistribution is feasible inside groups and coalitions. In [12], the authors show that even very limited (uni-dimentional) space of compensation schemes can guarantee coalitional stability in country formation model formally equivalent to UFLP special case on the line. However, a natural assumption for a variety of applications is that information about addresses of players is hidden and therefore no redistribution is possible; the only cost allocation rule available in such circumstances is a *equal-sharing* rule.

It is this rule under which [1] conduct their analysis. Adopting a model with the continuum of players uniformly distributed along the real line, authors show the emergence of coalitional stability. They also conjecture that stability is preserved in the general case for an arbitrary population distribution.

Some evidence in favor of this conjecture is contained in [9] where it is shown that, if only *connected* coalitions are able to pose a threat of secession, stability is assured. However in [4] we showed that the conjecture of [1] in fact does not hold: two counter-examples which are presented above, were constructed. This paper is a full-stop in the quest for coalitional stability under the equal-share and median rules.

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